

Comment on “Quantum secret sharing based on reusable Greenberger-Horne-Zeilinger states as secure carriers” [Phys. Rev. A **67**, 044302 (2003)]

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In a recent paper [S. Bagherinezhad and V. Karimipour, Phys. Rev. A **67**, 044302 (2003)], a quantum secret sharing protocol based on reusable GHZ states was proposed. However, in this Comment, it is shown that this protocol is insecure if Eve employs a special strategy to attack.

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In a recent paper [1], Bagherinezhad and Karimipour proposed a quantum secret sharing protocol based on reusable GHZ states. The security against both intercept-resend strategy and entangle-ancilla strategy was proved. However, here we will show that with the help of her ancilla Eve can obtain half of the data bits without being detected by the communication parties.

For convenience, we use the same notations as in Ref. [1]. We will begin with the property of states $|\bar{0}\rangle_{12}$ and $|\bar{1}\rangle_{12}$ under the Pauli operation $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (i.e., flipping). That is, it has the same effect to perform σ_x on the qubit 1 or 2 in the Bell states $|\bar{0}\rangle_{12}$ and $|\bar{1}\rangle_{12}$, i.e., the state $|\bar{0}\rangle_{12}$ ($|\bar{1}\rangle_{12}$) will be converted into $|\bar{1}\rangle_{12}$ ($|\bar{0}\rangle_{12}$) when a σ_x operation is performed on either of the two qubits. This can be described by

$$\begin{aligned}\sigma_{x1} \otimes \mathbf{I}_2 |\bar{0}\rangle_{12} &= \mathbf{I}_1 \otimes \sigma_{x2} |\bar{0}\rangle_{12} = |\bar{1}\rangle_{12} \\ \sigma_{x1} \otimes \mathbf{I}_2 |\bar{1}\rangle_{12} &= \mathbf{I}_1 \otimes \sigma_{x2} |\bar{1}\rangle_{12} = |\bar{0}\rangle_{12}\end{aligned}\quad (1)$$

where \mathbf{I} denotes the identity operation. As a result, the states $|\bar{0}\rangle_{12}$ and $|\bar{1}\rangle_{12}$ will be changed into each other under odd times σ_x operation, and be left alone under even times σ_x operation (including the operations on either qubit). This property will be used later.

Now let us give an explicit description of Eve's strategy. In the beginning, Alice, Bob and Charlie share a GHZ state $|G\rangle = (1/\sqrt{2})(|000\rangle + |111\rangle)$ as carrier, and the data bits Alice wants to distributed to Bob and Charlie can be represented by $q_1, q_2, q_3, \dots, q_n$. Eve prepares a qubit in state $|0\rangle$ as her ancilla.

(i) In the first round, Eve intercepts the first sending qubit (the one denoted by subscript 1) and performs a CNOT operation C_{1e} on this qubit and her ancilla after Alice sent the particles, which produces the state

$$|\Psi_{abce12}^0\rangle = \frac{1}{\sqrt{2}}(|000000\rangle + |111111\rangle) \quad (q_1 = 0)$$

$$\text{or } |\Psi_{abce12}^1\rangle = \frac{1}{\sqrt{2}}(|000111\rangle + |111000\rangle) \quad (q_1 = 1) \quad (2)$$

and then resends it to Bob. Here we use superscripts 0 and 1 to denote the states corresponding to $q_1 = 0$ and

$q_1 = 1$, respectively. This notation also applies to the following equations and we will, for simplicity, suppress the word “or” later.

It is obvious that these eavesdropping actions introduce no error when Bob and Charlie disentangle the sending qubits from the carrier. After that, the state of Alice, Bob, Charlie and Eve can be specified by

$$\begin{aligned}|\Theta_{abce}^0\rangle_{\text{odd}} &= \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \\ |\Theta_{abce}^1\rangle_{\text{odd}} &= \frac{1}{\sqrt{2}}(|0001\rangle + |1110\rangle)\end{aligned}\quad (3)$$

(ii) Eve performs a Hadamard gate on her ancilla. According to the protocol in Ref. [1], before entangle $|\bar{q}_2\rangle$ to the carrier, Alice, Bob and Charlie will do the same actions on their respective qubits. As a result, the entangled state will be converted into

$$\begin{aligned}|\Theta_{abce}^0\rangle_{\text{even}} &= H^{\otimes 4} |\Theta_{abce}^0\rangle_{\text{odd}} = \frac{1}{2\sqrt{2}}(|0000\rangle + |0011\rangle \\ &+ |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle) \\ |\Theta_{abce}^1\rangle_{\text{even}} &= H^{\otimes 4} |\Theta_{abce}^1\rangle_{\text{odd}} = \frac{1}{2\sqrt{2}}(|0000\rangle - |0011\rangle \\ &- |0101\rangle + |0110\rangle - |1001\rangle + |1010\rangle + |1100\rangle - |1111\rangle)\end{aligned}\quad (4)$$

(iii) In the second round, Eve intercepts the first sending qubit and performs a CNOT operation C_{e1} on this qubit and her ancilla after Alice sent the particles, and then resends it to Bob. By these actions, Eve's aim is to avoid being detected by the communication parties.

After Bob and Charlie disentangled the sending qubits from the carrier, according to the protocol in Ref. [1], each of the four parties has made a CNOT operation on their respective qubit (as the control qubit) and one of the sending qubits (as the target qubit). Furthermore, it can be seen from Eq. (4) that all the items in both $|\Theta_{abce}^0\rangle_{\text{even}}$ and $|\Theta_{abce}^1\rangle_{\text{even}}$ have even weights. Therefore, the effect of the above four CNOT operations is equivalent to flipping either qubit in the transmitted state $|\bar{q}\rangle$ even times. With the help of the property of $|\bar{q}\rangle$ introduced in above paragraphs, we can draw a conclusion

that Eve's actions introduce no error in this round and the state $|\Theta_{abce}^0\rangle_{\text{even}}$ or $|\Theta_{abce}^1\rangle_{\text{even}}$ is not changed. Hence Eve will avoid the detection of the communication parties, though she can not obtain any information about the data bit transmitted in this round.

(iv) As Alice, Bob and Charlie will do, Eve also performs a Hadamard gate on her ancilla. These four Hadamard operations change the entangled state $|\Theta_{abce}^0\rangle_{\text{even}}$ ($|\Theta_{abce}^1\rangle_{\text{even}}$) into $|\Theta_{abce}^0\rangle_{\text{odd}}$ ($|\Theta_{abce}^1\rangle_{\text{odd}}$) as described in Eq. (3), which will be used as a carrier in the next round.

(v) In the third round, Eve can obtain partial information about the data bit without being detected by performing operation C_{e1} , making a measurement on the first sending qubit and performing C_{e1} again.

The particular process is as follows. Suppose the classical bit to be transmitted (that is, q_3) is encoded as $|q, q\rangle$ ($q = 0$ or 1). When Alice has entangled the sending qubits to the carrier, the state of whole system can be described by

$$\begin{aligned} |\Phi_{abce12}^0\rangle &= \frac{1}{\sqrt{2}}(|0000, q, q\rangle + |1111, q+1, q+1\rangle) \\ |\Phi_{abce12}^1\rangle &= \frac{1}{\sqrt{2}}(|0001, q, q\rangle + |1110, q+1, q+1\rangle) \end{aligned} \quad (5)$$

where the addition is performed modulo 2.

After the two particles were sent out by Alice, Eve intercepts the first qubit and performs a CNOT operation C_{e1} , producing the state

$$\begin{aligned} |\Omega_{abce21}^0\rangle &= \frac{1}{\sqrt{2}}(|0000, q\rangle + |1111, q+1\rangle)_{abce2}|q\rangle_1 \\ |\Omega_{abce21}^1\rangle &= \frac{1}{\sqrt{2}}(|0001, q\rangle + |1110, q+1\rangle)_{abce2}|q+1\rangle_1 \end{aligned} \quad (6)$$

where we have suppressed the \otimes symbol. It can be seen from Eq. (6) that the qubit 1 has been disentangled from the entangled state. Eve then makes a measurement on this qubit in $B_z = \{|0\rangle, |1\rangle\}$ basis and gets the result q (when $q_1 = 0$) or $q+1$ (when $q_1 = 1$). Let r_i ($i = 1, 2, \dots, n$) denote the eavesdropping results on data bits q_i . Eve knows that $r_3 = q_3$ (when $q_1 = 0$) or $r_3 = q_3 + 1$ (when $q_1 = 1$). Afterwards, Eve performs C_{e1} again to recover the state as in Eq. (5), and then resends this particle to Bob.

It can be easily verified that Eve's actions introduce no error when Bob and Charlie disentangle the sending particles from the carrier. Besides, the carrier and Eve's ancilla is still in the state $|\Theta_{abce}^0\rangle_{\text{odd}}$ or $|\Theta_{abce}^1\rangle_{\text{odd}}$ as in Eq. (3).

(vi) Eve uses similar operations to eavesdrop in the following rounds. That is, she takes the same actions as

in step.(v) in odd rounds and the actions as in step.(iii) in even rounds. Furthermore, when Alice, Bob and Charlie perform Hadamard gates on their respective qubits at the end of every round, Eve does the same thing (as in step.(ii) and step.(iv)).

Through above strategy, Eve can avoid the detection of the three legal parties and even extract information about the odd numbered data bits at the end of communication. We mean the measurement results $r_3, r_5, r_7, \dots, r_{2m+1}$ (where $m = 1, 2, \dots$ and $2m+1 \leq n$). Thus Eve can draw a conclusion that Alice's odd numbered data bits $q_1, q_3, q_5, \dots, q_{2m+1}$ must be equal to 0, $r_3, r_5, \dots, r_{2m+1}$ or 1, $r_3 + 1, r_5 + 1, \dots, r_{2m+1} + 1$.

The above eavesdropping result seems not so beautiful because there are still two possibilities. However, according to the protocol in Ref. [1], the three legal parties have to compare a subsequence of the data bits publicly to detect eavesdropping, which will leak useful information to Eve. More specifically, as long as any odd numbered data bit is announced, Eve can determine which of the two possible results is true. By this means Eve can obtain the odd numbered data bits completely except for the little-probability event that all the compared bits are even numbered.

It should be pointed out that the QKD protocol in Ref. [2], which inspires the work of Bagherinezhad and Karimipour [1], has the same hidden troubles, where Eve can use similar strategy to extract partial information about the distributed key. From this kind of attack we can draw two instructive conclusions. Firstly, when we employ the maximally entangled states as carriers like the protocols in [1, 2], it can not effectively prevent a potential eavesdropping to perform a Hadamard gate [1] or a $\pi/4$ rotation [2] on each qubit in the carriers. Secondly, scheme designers should ensure that the announcement of subsequence, i.e., the means by which the users detect eavesdropping, would leak no useful information to the eavesdropper.

In conclusion, we have presented a eavesdropping strategy which allows Eve to obtain half of the data bits without being detected by the communication parties in Bagherinezhad-Karimipour protocol. Consequently this protocol is insecure against this type of attack.

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[1] S. Bagherinezhad and V. Karimipour, Phys. Rev. A **67**, 044302 (2003).

[2] Y. Zhang, C. Li and G. Guo, Phys. Rev. A **64**, 024302 (2001).